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Some limit relations in the A_1 tableau of Dunkl-Cherednik operators

Dedicated to Tom Koornwinder on the occasion of his 60th birthday

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ABSTRACT

In this paper we study some limit relations involving some q-special functions related with the A_1 (root system) tableau of Dunkl-Cherednik operators. Concretely we consider the limits involving the nonsymmetric q-ultraspherical polynomials (q-Rogers polynomials), ultraspherical polynomials (Gegenbauer polynomials), q-Hermite and Hermite polynomials.

1. INTRODUCTION

In this paper we study some limit relations involving some q-special functions related with the A_1(root system) tableau of Dunkl-Cherednik operators. In fact we will continue the research initiated by Cherednik on the interconnection of the A_1 root system and the special functions. In the paper [14] Koornwinder proved that the q-ultraspherical polynomials are closely related with the extended affine Hecke Algebra of type A_1 and starting from them it is shown how several other families of q-special functions are related to this Hecke algebra. It is important to notice that the q-ultraspherical polynomials are instances of the celebrated Askey-Wilson polynomials for which the Hecke algebra approach also works, for more details on this and further references see the very recent paper [16]. In his paper [14] Koornwinder proved several limit relations involving the q-ultraspherical polynomials, ultraspherical polynomials and func-

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tions, the Hall-Littlewood polynomials and the Bessel functions. Here we will complete it studying the limit relations involving the q-ultraspherical polynomials, the q-Hermite and the Hermite polynomials. Let us point here that we are considering the limit transitions of nonsymmetric analogues of the special families of orthogonal polynomials, the limit transitions for the usual families are well known [12].

For an introduction of a q-special functions see e.g. [7,12] and for a review of the theory of root systems see [10]. The connection of root system with hypergeometric functions was studied in details in [9,8] (see also the survey [13] for a detailed introduction). The special functions associated to a root system have an algebraic interpretation [1,2,3] in the framework of affine and graded Hecke algebras [15] (see also [11]). Of particular interest is the paper [16] where the connection between the Askey-Wilson polynomials and the non-reduced affine root system of rank one is established and it is used for deriving several properties of the Askey-Wilson polynomials. The theory of Dunkl operators is also used to introduce the multi-variable orthogonal polynomials (see [17] for the Hermite case and the nice book [5]) and for the connection with Bessel functions we refer to [4].

The structure of the paper is the following. In Section 2 we include some definitions which will be useful in the next sections. In Section 3, some previous results on q-ultraspherical polynomials are presented which allow in Section 4 to obtain the limit to q-Hermite polynomials. Finally, in Sections 5 and 6 the limit from q-Hermite polynomials to Hermite polynomials and ultraspherical polynomials to Hermite polynomials are considered. In the last two cases it was necessary to work in the space of vector functions instead of the standard space of Laurent polynomials.

2. PRELIMINARIES

Let V be a d-dimensional real vector space with an inner product $\langle\cdot\,,\,\cdot\rangle$. For $\alpha\in V\backslash\{0\}$ let s_α denotes the orthogonal reflection with respect to the hyperplane orthogonal to α:

$$s_\alpha(\beta)=\beta-\frac{2\langle\beta,\,\alpha\rangle}{\langle\alpha,\,\alpha\rangle}\alpha,\qquad(\beta\in V).$$

A root system in V is a finite subset R of $V\backslash\{0\}$ which spans V and satisfies

$$s_\alpha(\beta)\in R\quad\text{and}\quad\frac{2\langle\beta,\,\alpha\rangle}{\langle\alpha,\,\alpha\rangle}\in\mathbb{Z}\quad\forall\alpha,\,\beta\in R.$$

A root system R that satisfies the condition: if $\alpha,\,\beta\in R$ and $\alpha=c\beta$ for some $c\in\mathbb{R}$, then $c=\pm1$, is called a reduced root system.

We will deal here with the *irreducible* root system A_1 which corresponds to the case $d=1$ and $R=\{\pm2\}\subset\mathbb{R}$. It is represented in figure 2

Fig. 1: Root System A_1.

Here we will use the standard notation for the q-special functions [7]. Throughout the paper we will suppose that $q \in (0,1)$. For given $q \in (0,1)$ the q-basic hypergeometric series ${}_r\varphi_q$ is defined by

$$(1)\qquad {}_r\varphi_p\left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_p \end{matrix}; q, z\right) = \sum_{k=0}^{\infty} \frac{(a_1;q)_k \cdots (a_r;q)_k}{(b_1;q)_k \cdots (b_p;q)_k} \frac{z^k}{(q;q)_k} \left[(-1)^k q^{k/2(k-1)}\right]^{p-r+1},$$

where, for $k = 0, 1, 2,,$

$$(2)\qquad (a;q)_0 := 1, \quad (a;q)_k := \prod_{m=0}^{k-1} (1 - aq^m), \quad (a;q)_\infty = \prod_{m=0}^{\infty} (1 - aq^m).$$

From (1) it is clear that for $a_1 = q^{-n}$ with $n = 0, 1, 2, ..$ the series terminates after $k = n$ terms. Also we need the generalized hypergeometric function ${}_p\mathrm{F}_q$

$$(3)\qquad {}_p\mathrm{F}_q\left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix}\Big| x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!},$$

where $(a)_k$ is the Pocchammer symbol

$$(4)\qquad (a)_0 := 1, \ (a)_k := a(a+1)(a+2)\cdots(a+k-1), \ k = 1, 2, 3, \dots \ .$$

Notice that for $a_1 = -n$ with $n = 0, 1, 2, ..$ the series also terminates after $k = n$ terms.

Let us now define the hypergeometric polynomials which will be considered in this paper. We start with the q-ultraspherical polynomials which are a particular case of the Askey-Wilson polynomials [12]. They are the symmetric Laurent polynomials

$$(5)\qquad R_n^{k,q}(z) = {}_4\varphi_3\left(\begin{matrix} q^{-n/2}, q^{n/2+k}, q^{k/2}z, q^{k/2}z^{-1} \\ -q^k, q^{k/2+\frac{1}{4}}, -q^{k/2+\frac{1}{4}} \end{matrix}; q^{1/2}, q^{1/2}\right), \quad k \in \mathbb{R}.$$

Notice that $R_n^{k,q}(z)$ are polynomials of degree n in $\frac{z+z^{-1}}{2}$. Here we are using the notation introduced in [14] (for the standard notation see e.g. [7,12]). Let us point out that the $R_n^{k,q}(z)$ polynomials are related with the "standard" q-ultraspherical (Rogers) polynomials [12, Eq. (3.10.15)] by the expression

$$(6)\qquad R_n^{k,q}(z) = \frac{\beta^{n/2}(q;q)_n}{(\beta^2;q)_n} C_n(\cos\theta;\beta|q), \qquad \beta = q^k,$$

which follows from the Singh transformation formula [7, pag. 89].

The polynomials $R_n^{k,q}(z)$ are orthogonal with respect to a bilinear (positive definite in the space of Laurent polynomials with real coefficients) form $\langle F, G\rangle_{k,q}$ defined by

$$\langle F,G\rangle_{k,q}=\frac{1}{2\pi i}\oint_{(0)}F(z)G(z^{-1})\frac{(z^2;q)_\infty(z^{-2};q)_\infty}{(q^k z^2;q)_\infty(q^k z^{-2};q)_\infty}\frac{dz}{z}, \tag{7}$$

and their norms are given by

$$\langle R_n^{k,q}, R_m^{k,q}\rangle_{k,q}=\frac{2(1-q^k)q^{kn}(q;q)_n(q^k;q)_\infty(q^{k+1};q)_\infty}{(1-q^{k+n})(q^{2k};q)_n(q^{2k};q)_\infty(q;q)_\infty}\delta_{nm}.$$

The ultraspherical polynomials are given by

$$R_n^k(z)={}_2\mathrm{F}_1\left[\begin{matrix}-n\ ,n+2k\\ k+1/2\end{matrix};\frac{1}{2}\left(1-\frac{z+z^{-1}}{2}\right)\right]. \tag{8}$$

They satisfy the orthogonality relation

$$\langle R_n^k,R_m^k\rangle_k\equiv\int_0^\pi R_n^k(e^{i\theta})R_m^k(e^{i\theta})\sin^{2k}\theta\,d\theta=\frac{\sqrt{\pi}\Gamma(k+1/2)}{(n+k)\Gamma(k)}\delta_{nm}. \tag{9}$$

We want to point out here that

$$\int_0^\pi R_n^k(e^{i\theta})R_m^k(e^{i\theta})\sin^{2k}\theta\,d\theta=\int_{-1}^1 R_n^k(x)R_m^k(x)(1-x^2)^{k-1/2}dx.$$

The ultraspherical polynomials are related with the q-ultraspherical polynomials (5) by the limit relation

$$R_n^k(z)=\lim_{q\to 1-}R_n^{k,q}(z). \tag{10}$$

The continuous q-Hermite polynomials are defined by

$$H_n^q(z)=z^n{}_2\varphi_0\left(\begin{matrix}q^{-n}\ ,0\\ \text{—}\end{matrix};q,q^nz^{-2}\right), \tag{11}$$

and satisfy the orthogonality property

$$\langle H_n^q, H_m^q\rangle_q=\frac{1}{2\pi i}\oint_{(0)}H_n^q(z)H_m^q(z^{-1})(z^2;q)_\infty(z^{-2};q)_\infty\frac{dz}{z}=\frac{2}{(q^{n+1};q)_\infty}\delta_{nm}. \tag{12}$$

They also can be obtained as a limit case of the q-ultraspherical polynomials (5)

$$H_n^q(z)=\lim_{k\to\infty}q^{-1/2nk}R_n^{k,q}(z). \tag{13}$$

The above formula follows from the Eq. (6) and the formula (4.10.3) of [12].

Finally, we will consider the Hermite polynomials defined by

$$H_n(x)=(2x)^n{}_2\mathrm{F}_0\left(\begin{matrix}-n/2\ ,-(n-1)/2\\ \text{—}\end{matrix};-\frac{1}{x^2}\right), \tag{14}$$

which satisfy the orthogonality property

$$(15)\qquad \langle H_n, H_m\rangle = \int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx = 2^n n!\sqrt{\pi}\delta_{nm}.$$

They can be obtained as a limit case of the q-Hermite polynomials (14) [12, Eq. (5.26.1)]

$$(16)\qquad H_n(x) = \lim_{q\to 1-} \frac{H_n^q\left(\sqrt{\frac{1-q}{2}}x\right)}{\left(\frac{1-q}{2}\right)^{n/2}},$$

or as a limit case of the ultraspherical polynomials (8) [12, pag. 69]

$$(17)\qquad 2^{-n}H_n(x) = \lim_{k\to\infty} k^{n/2}R_n^k\left(\frac{x}{\sqrt{k}}\right).$$

3. NONSYMMETRIC q-ULTRASPHERICAL POLYNOMIALS AND CHEREDNIK OPERATORS.

Let us introduce the operators T, Y^- and Y^+ depending on a parameter $t = q^{-k/2}$ and acting on Laurent polynomials as follows

$$(18)\qquad (TF)(z) = tF(z^{-1}) + (t-t^{-1})\frac{F(z)-F(z^{-1})}{1-z^2},$$

$$(19)\qquad (Y^+F)(z) = tF(q^{1/2}z) + (t-t^{-1})\frac{F(q^{1/2}z)-F(q^{-1/2}z^{-1})}{q^{-1}z^{-2}-1},$$

$$(20)\qquad (Y^-F)(z) = t^{-1}F(q^{-1/2}z) + (t-t^{-1})\frac{F(q^{-1/2}z)-F(q^{-1/2}z^{-1})}{z^{-2}-1}.$$

They are usually called generalized reflection (T) and q-difference-reflection ($Y^\pm$) operators and they are the so-called Cherednik operators. Notice that $F(z) = F(z^{-1})$ (i.e., F is symmetric) iff $TF = tF$. The operators T and $Y^\pm$ obey the following commutation relations

$$(21)\qquad \begin{aligned} &(T-t)(T+t^{-1}) = 0, \quad TY^+ - Y^-T = (t-t^{-1})Y^+,\\ &Y^+Y^- = Y^-Y^+ = I, \end{aligned}$$

where I denotes the identity operator. The first relation is known as *Hecke relation* as well as the algebra generated by these three operators is called the extended affine Hecke algebra of type A_1 (see e.g. [13]).

The eigenfunctions of the operators $Y^\pm$ are the so-called nonsymmetric q-ultraspherical polynomials [14] defined by

$$(22)\quad E_w^{k,q}(z) = R_n^{k,q}(z) + \frac{q^{k+1/2}(q^{1/2(w-k)} - q^{1/2(k-w)})(q^{-k/2}z - q^{k/2}z^{-1})}{(1+q^k)(1-q^{2k+1})}R_{n-1}^{k+1,q}(z),$$

for $w = \pm(n+k)$, $n = 1, 2, \dots$, i.e.,

$$(23)\qquad Y^+E_w^{k,q}(z) = q^{-w/2}E_w^{k,q}(z) \quad\text{and}\quad Y^-E_w^{k,q}(z) = q^{w/2}E_w^{k,q}(z).$$

Notice [14] that the operator $Y^+ + Y^-$ acting on any symmetric Laurent polynomial F becomes into a q–difference operator

$$(Y^+ + Y^-)F(z) = q^{-k/2}\left(\frac{1-q^kz^2}{1-z^2}F(q^{1/2}z) + \frac{1-q^kz^{-2}}{1-z^{-2}}F(q^{-1/2}z)\right),$$

then, using (23) we can recover [14] the eigenvalue equation for the q-ultraspherical polynomials

$$\frac{1-q^kz^2}{1-z^2}R_n^{k,q}(q^{1/2}z) + \frac{1-q^kz^{-2}}{1-z^{-2}}R_n^{k,q}(q^{-1/2}z) = (q^{-n/2} + q^{n/2+k})R_n^{k,q}(z).$$

The nonsymmetric q-ultraspherical polynomials satisfy a biorthogonality relation with respect to a bilinear form on the space of Laurent polynomials [14] defined as

$$\text{(24)} \qquad \langle F, G\rangle'_{k,q} = \frac{1}{2\pi i}\oint F(z)G(z^{-1})\frac{(qz^2;q)_\infty(z^{-2};q)_\infty}{(q^{k+1}z^2;q)_\infty(q^kz^{-2};q)_\infty}\frac{dz}{z}.$$

More concretely,

$$\langle E_w^{k,q}, E_v^{k,q^{-1}}\rangle'_{k,q} = \delta_{wv}\begin{cases} \dfrac{(1-q^{2k})(1+q^k)q^{k(n-1)}(q;q)_n(q^k;q)_\infty(q^{k+1};q)_\infty}{(1-q^n)(q^{2k};q)_n(q^{2k};q)_\infty(q;q)_\infty}, & w=-n-k \\ \dfrac{(1-q^{2k})(1+q^k)q^{kn}(q;q)_n(q^k;q)_\infty(q^{k+1};q)_\infty}{(1-q^{2k+n})(q^{2k};q)_n(q^{2k};q)_\infty(q;q)_\infty}, & w=n+k. \end{cases}$$

In [14] the limits of nonsymmetric q-ultraspherical polynomials to nonsymmetric ultraspherical polynomials, nonsymmetric q-ultraspherical polynomials to nonsymmetric ultraspherical functions, nonsymmetric q-ultraspherical polynomials to Hall-Littlewood polynomials among others have been successfully studied (in details). Here we will consider other limits which are also interesting and which have not been considered there. In such a way we will complete the A_1 classification of q-basic polynomials.

4. LIMIT NONSYMMETRIC q-ULTRASPHERICAL POLYNOMIALS TO NONSYMMETRIC q-HERMITE POLYNOMIALS

Let us consider the limit when $k \to \infty$ ($q^k \to 0$). We will define the operators X^+, X^{-1} and $\tilde{T}$ in the following way

$$\text{(25)} \qquad X^+ := \lim_{k\to\infty}[q^{k/2}Y^+], \quad X^- := \lim_{k\to\infty}[q^{k/2}Y^-], \quad \tilde{T} := \lim_{k\to\infty}[q^{k/2}T].$$

This leads to

$$\text{(26)} \qquad (\tilde{T}F)(z) = F(z^{-1}) + \frac{F(z) - F(z^{-1})}{1-z^2},$$

(27) $$(X^+ F)(z) = F(q^{1/2}z) + \frac{F(q^{1/2}z) - F(q^{-1/2}z^{-1})}{q^{-1}z^{-2} - 1},$$

(28) $$(X^- F)(z) = \frac{F(q^{-1/2}z) - F(q^{-1/2}z^{-1})}{1 - z^{-2}},$$

and they satisfy the following commutation relation

(29) $$\tilde{T}^2 = \tilde{T}, \quad \tilde{T}X^+ - X^-\tilde{T} = X^+, \quad X^+X^- = X^-X^+ = 0.$$

Let define the functions $E^q_{\pm n}(z)$ as follows

(30) $$\begin{aligned} E^q_n(z) &= \lim_{q^k \to 0} q^{-1/2nk} E^{k,q}_{n+k}(z) = H^q_n(z) + (q^n - 1)zH^q_{n-1}(z), \\ E^q_{-n}(z) &= \lim_{q^k \to 0} q^{-1/2nk+k} E^{k,q}_{-n-k}(z) = zH^q_{n-1}(z), \end{aligned}$$

where $x = \frac{z+z^{-1}}{2}$ and $n = 1, 2, 3, \ldots$. Then, from relations (23) by taking the limit $q \to 1-$ we find

(31) $$X^+E^q_n(z) = q^{-n/2}E^q_n(z), \quad X^+E^q_{-n}(z) = 0,$$

(32) $$X^-E^q_{-n}(z) = q^{-n/2}E^q_{-n}(z), \quad X^-E^q_n(z) = 0.$$

The operators X^+, X^- and $\tilde{T}$ will act on the space $\mathrm{Span}\{E^q_n, E^q_{-n}\}$. A function in this space is symmetric ($F(z) = F(z^{-1})$) iff it is proportional to the q–Hermite polynomials $H^q_n(z)$. This, joint with the definition of $\tilde{T}$, means that the q–Hermite polynomials are the eigenfunctions of $\tilde{T}$. Notice also that applying the operator $X^+ + X^-$ to a symmetric polynomial $F(z)$ we find

$$(X^+ + X^-)F(z) = \frac{1}{1 - z^2}F(q^{1/2}z) + \frac{1}{1 - z^{-2}}F(q^{-1/2}z),$$

then by using (31)–(32) we recover the eigenvalue equation for the q–Hermite polynomials $H^q_n(z)$ [12, §3.26]

$$\frac{1}{1 - z^2}H^q_n(q^{1/2}z) + \frac{1}{1 - z^{-2}}H^q_n(q^{-1/2}z) = q^{-n/2}H^q_n(z)$$

Finally, let us consider the orthogonality properties of the functions E^q_n, E^q_{-n}. We can do this in two different ways: 1- calculating directly the norms, 2- taking limits in the expressions (24). Both leads us to the following result: The system E^q_n, E^q_{-n} satisfy the following biorthogonality relation (compare with (12)) $(n = 0, 1, \ldots)$

(33) $$\langle E^q_k, \tilde{E}^q_l \rangle'_q = \frac{1}{2\pi i}\oint_{(0)} E^q_k(z)\tilde{E}^q_l(z^{-1})(qz^2; q)_\infty(z^{-2}; q)_\infty \frac{dz}{z} = \delta_{kl}d_k, \quad k, l \in \mathbb{Z},$$

where the dual system $\tilde{E}^q_n$, $\tilde{E}^q_{-n}$ is defined as follows

$$
(34)\qquad
\begin{aligned}
\widetilde{E}^q_n(z) &= \lim_{q^k\to 0} q^{1/2nk} E^{k,q^{-1}}_{n+k}(z) = H^q_n(z),\\
\widetilde{E}^q_{-n}(z) &= \lim_{q^k\to 0} q^{1/2nk} E^{k,q^{-1}}_{-n-k}(z) = H^q_n(z) - z^{-1}H^q_{n-1}(z),
\end{aligned}
$$

and the norms are given by

$$
d_n = \frac{1}{(q^{n+1};q)_\infty}, \quad d_{-n} = \frac{1}{(q^n;q)_\infty}.
$$

The above result (33) can be found directly from the fact that $(q^{n+1};q)_\infty/(q^n;q)_\infty = (1-q^n)^{-1}$ and the following straightforward lemma.

Lemma 4.1. *Let $\langle F, G\rangle_q$ and $\langle F, G\rangle'_q$ be the following two bilinear forms on the space of Laurent polynomials*

$$
\begin{aligned}
\langle F, G\rangle_q &= \frac{1}{2\pi i}\oint_{(0)} F(z)G(z^{-1})(z^2;q)_\infty(z^{-2};q)_\infty \frac{dz}{z}\\
\langle F, G\rangle'_q &= \frac{1}{2\pi i}\oint_{(0)} F(z)G(z^{-1})(qz^2;q)_\infty(z^{-2};q)_\infty \frac{dz}{z},
\end{aligned}
$$

and suppose that $f(z)$ and $g(z)$ are symmetric functions (a function f is a symmetric function if $f(z) = f(z^{-1})$). Then, the following relations hold

(a) $\langle zf, zg\rangle'_q = \langle f, g\rangle'_q = \frac{1}{2}\langle f, g\rangle_q$, (b) $\langle f, zg\rangle'_q = \langle zf, g\rangle_q$ (c) $\langle zf, g\rangle'_q = 0$,
(d) $\langle f, gz^{-1}\rangle'_q = 0$, (e) $\langle zf, z^{-1}g\rangle'_q = -\frac{1}{2}\langle f, g\rangle_q$.

5. LIMIT NONSYMMETRIC q-HERMITE POLYNOMIALS TO HERMITE POLYNOMIALS

Let us consider the limit nonsymmetric q-Hermite polynomials to Hermite polynomials. Since the classical limit (16) involves functions on x and in (30) the functions depend explicitly on z we are obliged to use, instead scalar functions $F(z)$, vector functions of the form $\mathbf{f} := (f_1(x), f_2(x))^T$, where, $x = (z + z^{-1})/2$. They are related to each other by formulas

$$
(35)\qquad
\begin{aligned}
&F(z) = f_1(x) + (z - z^{-1})f_2(x),\\
&f_1(x) = \frac{F(z) + F(z^{-1})}{2}, \quad f_2(x) = \frac{F(z) - F(z^{-1})}{2(z - z^{-1})}.
\end{aligned}
$$

Notice that f_1 and f_2 are symmetric functions on z.

The next step is to rewrite the action of operators X^+, X^- acting on the space of Laurent polynomials $F(z)$ as the equivalent matrix-operators $\mathbf{X}^+$ and $\mathbf{X}^-$ acting on the space of vector functions $(f_1(x), f_2(x))^T$. Using the definition (26)–(28) as well as (35) we find that

$$
X^+ \sim \mathbf{X}^+ = \begin{pmatrix} X^+_{11} & X^+_{12}\\ X^+_{21} & X^+_{22}\end{pmatrix},
$$

where

$$X_{11}^{+}f_1(x)=\frac{1}{2}\left[f_1\left(\frac{q^{1/2}z+q^{-1/2}z^{-1}}{2}\right)+f_1\left(\frac{q^{-1/2}z+q^{1/2}z^{-1}}{2}\right)\right],$$

$$X_{12}^{+}f_2(x)=-\left(\frac{q^{1/2}z+q^{-1/2}z^{-1}}{2}\right)f_2\left(\frac{q^{1/2}z+q^{-1/2}z^{-1}}{2}\right)$$
$$-\left(\frac{q^{-1/2}z+q^{1/2}z^{-1}}{2}\right)f_2\left(\frac{q^{-1/2}z+q^{1/2}z^{-1}}{2}\right),$$

$$X_{21}^{+}f_1(x)=\frac{f_1\left(\frac{q^{1/2}z+q^{-1/2}z^{-1}}{2}\right)-f_1\left(\frac{q^{-1/2}z+q^{1/2}z^{-1}}{2}\right)}{2(z-z^{-1})},$$

$$X_{22}^{+}f_2(x)=\frac{-\left(\frac{q^{1/2}z+q^{-1/2}z^{-1}}{2}\right)f_2\left(\frac{q^{1/2}z+q^{-1/2}z^{-1}}{2}\right)+\left(\frac{q^{-1/2}z+q^{1/2}z^{-1}}{2}\right)f_2\left(\frac{q^{-1/2}z+q^{1/2}z^{-1}}{2}\right)}{z-z^{-1}}.$$

We are interested to find the eigenfunctions of the operator $\mathbf{X}^{+}$. Using Eq. (35) and (30) we find that

$$\text{(36)}\qquad E_n^q(z)\longmapsto\begin{pmatrix}H_n^q(x)+(q^n-1)xH_{n-1}^q(x)\\ \frac{q^n-1}{2}\,H_{n-1}^q(x)\end{pmatrix},\quad E_{-n}^q(z)\longmapsto\begin{pmatrix}xH_{n-1}^q(x)\\ \frac{1}{2}H_{n-1}^q(x)\end{pmatrix},$$

where $z=e^{-i\theta}$. Notice that with our notation $\frac{q^{1/2}z+q^{-1/2}z^{-1}}{2}=\cos(\theta-i\log\sqrt{q})$ and $\frac{q^{-1/2}z+q^{1/2}z^{-1}}{2}=\cos(\theta+i\log\sqrt{q})$, $x=\cos\theta$. In order to take limits (and obtain non-trivial relations) we will consider the operator $\mathbf{X}^{+}-\mathrm{I}$. Let $\omega=\sqrt{\frac{1-q}{2}}$, i.e., $q=1-2\omega^2$, and let us suppose that the following limits exist

$$\text{(37)}\qquad \lim_{\omega\to 0}\frac{f_1(\omega x)}{\omega^n}=\tilde{f}_1(x),\quad \lim_{\omega\to 0}\frac{f_2(\omega x)}{\omega^{n+1}}=\tilde{f}_2(x).$$

Let us consider the case of the vector function corresponding to E_n^q. Notice that, from (16), in the case when f_1 and f_2 are given by the first Eq. in (36) the conditions (37) hold. Moreover $\tilde{f}_1(x)=H_n(x)$ and $\tilde{f}_2(x)=-nH_{n-1}(x)$. Then, the first component of the matrix equation equivalent to (31) becomes

$$\text{(38)}\qquad \frac{X_{11}^{+}-I}{\omega^2}f_1(\omega x)+\frac{X_{12}^{+}}{\omega^2}f_2(\omega x)=\frac{q^{-n/2}-1}{\omega^2}f_1(\omega x).$$

If we divide (38) by ω^n and take the limit when $\omega\to 0$ we obtain the equation

$$\text{(39)}\qquad -\frac{1}{2}\frac{d^2}{dx^2}\tilde{f}_1(x)-2x\tilde{f}_2(x)=n\tilde{f}_1(x),$$

or equivalently

$$-\frac{1}{2}\frac{d^2}{dx^2}H_n(x)+2xnH_{n-1}(x)=nH_n(x).$$

The last Eq. is equivalent to the second order differential equation satisfied by the Hermite polynomials since $2nH_{n-1}(x)=H_n'(x)$.

For the other equation (the second component of the matrix equation equivalent to (31)) we have

$$(40)\qquad \left[\frac{X^{+}_{21}}{\omega}\right]\frac{f_1(\omega x)}{\omega^n}+\left[X^{+}_{22}-I\right]\frac{f_2(\omega x)}{\omega^{n+1}}=(q^{-n/2}-1)\frac{f_2(\omega x)}{\omega^{n+1}}.$$

If we now take the limit $\omega \longrightarrow 0$, it becomes

$$(41)\qquad -\frac{1}{2}\frac{d}{dx}\tilde{f}_1(x)-\tilde{f}_2(x)=0,$$

which, by using the conditions (37), gives the identity

$$-\frac{1}{2}\frac{d}{dx}H_n(x)+nH_{n-1}(x)=0,\quad \text{or}\quad \frac{d}{dx}H_n(x)=2nH_{n-1}(x).$$

Next, we will consider the action of the operator $\mathbf{X}^{+}$ on the vector function corresponding to E^{q}_{-n}in (36). In this case we see that the conditions (37) do not hold. They should be changed by the following ones

$$(42)\qquad \lim_{\omega\to 0}\frac{f_1(\omega x)}{\omega^n}=\hat{f}_1(x),\quad \lim_{\omega\to 0}\frac{f_2(\omega x)}{\omega^{n-1}}=\hat{f}_2(x).$$

Then, the Eqs. equivalent to (38) and (40) lead to the Eqs.

$$\omega^2\left[\frac{X^{+}_{11}-I}{\omega^2}\right]\frac{f_1(x)}{\omega^n}+\left[\frac{X^{+}_{12}}{\omega}\right]\frac{f_2(\omega x)}{\omega^{n-1}}=-\frac{f_1(\omega x)}{\omega^n},$$
$$\omega^2\left[\frac{X^{+}_{21}}{\omega}\right]\frac{f_1(\omega x)}{\omega^n}+\left[X^{+}_{22}-I\right]\frac{f_2(\omega x)}{\omega^{n-1}}=-\frac{f_1(\omega x)}{\omega^{n-1}},$$

which in the limit $\omega \longrightarrow 0$ gives us the Eqs.

$$(43)\qquad -2x\hat{f}_2(x)=-\hat{f}_1(x),\quad -\hat{f}_2(x)=-\hat{f}_2(x),$$

respectively. A simple inspection on (36) (second formula) leads us to $\hat{f}_1(x)=xH_{n-1}(x)$ and $\hat{f}_2(x)=\frac{1}{2}H_{n-1}(x)$, so that the last formulas do not give any interesting result.

For the second operator $\mathbf{X}^{-}$ the procedure is analogously, in particular the conditions (37) and (42) should be imposed. We will here only give the resulting equations obtaining by taking the corresponding limits, i.e., the equations equivalent to (39), (41) and (43). They lead to the equations

$$H_n(x)=H_n(x),\quad \frac{d}{dx}H_n(x)=2nH_{n-1}(x),\quad xH_{n-1}(x)=xH_{n-1}(x),$$
$$\frac{d^2}{dx^2}H_{n-1}(x)-2x\frac{d}{dx}H_{n-1}(x)+2(n-1)H_{n-1}(x)=0,$$

respectively.

Let us now consider the orthogonality property. First we need to rewrite the orthogonality for the functions $E^{q}_{\pm n}$ (33) in the space of vector functions. Notice that

$$\langle F,G\rangle'_q=\langle f_1,g_1\rangle'_q+\langle (z-z^{-1})f_2,g_1\rangle'_q+\langle f_1,(z-z^{-1})g_2\rangle'_q$$
$$+\langle (z-z^{-1})f_2,(z-z^{-1})g_2\rangle'_q,$$

where F and G are given by formula (35).

Next we add the numbers $\langle -zf_2, g_1\rangle_q' = \langle f_1, z^{-1}g_2\rangle_q' = 0$ (see lemma (4.1)) and use the lemma (4.1) to find

$$\langle F, G\rangle_q' = \frac{1}{2}\langle f_1, g_1\rangle_q - \langle xf_2, g_1\rangle_q + \langle f_1, xg_2\rangle_q + 2\langle(1-x^2)f_2, g_2\rangle_q,$$

which, in matrix form, can be written as

$$\langle F, G\rangle_q' = \frac{1}{2\pi i}\oint_{(0)}(f_1(x), f_2(x))\begin{pmatrix}\rho(z)/2 & x\rho(z)\\ -x\rho(z) & 2(1-x^2)\rho(z)\end{pmatrix}\begin{pmatrix}g_1(x)\\ g_2(x)\end{pmatrix}\frac{dz}{z},$$

where $x = \frac{z+z^{-1}}{2}$ and $\rho(z) = (z^2;q)_\infty(z^{-2};q)_\infty$. Here the dual system of (36) is

$$\text{(44)}\qquad \widetilde{E}_n^q(z)\longmapsto\begin{pmatrix}H_n^q(x)\\ 0\end{pmatrix},\quad \widetilde{E}_{-n}^q(z)\longmapsto\begin{pmatrix}H_n^q(x) - xH_{n-1}^q(x)\\ \frac{1}{2}H_{n-1}^q(x)\end{pmatrix}.$$

Because of Eq. (36), the above expression can be written for E_n^q as

$$\int_{-1}^{1}(H_n^q(x) + (q^n-1)xH_{n-1}^q(x), \frac{q^n-1}{2}H_{n-1}^q(x))\begin{pmatrix}w(x)/2 & xw(x)\\ -xw(x) & 2(1-x^2)w(x)\end{pmatrix}\begin{pmatrix}H_m^q(x)\\ 0\end{pmatrix}dx$$

where

$$w(x) = \frac{\prod_{k=0}^{\infty}(1-2xq^k+q^{2k})(1+2xq^k+q^{2k})(1-2xq^{k+1/2}+q^{2k+1})(1+2xq^{k+1/2}+q^{2k+1})}{\sqrt{1-x^2}}.$$

As before, we will change $x\mapsto\omega x$, multiply by $s(\omega)\omega^{-n-m}$ and take the limits $\omega\to 0$ (the corresponding scaling factor $s(\omega)$ is such that $s(\omega)w(\omega x)\to e^{-x^2}$). Then we find ($n\neq m$)

$$\text{(45)}\qquad \int_{-\infty}^{\infty}(H_n(x), 0)\begin{pmatrix}\frac{1}{2}e^{-x^2} & 0\\ 0 & 2e^{-x^2}\end{pmatrix}\begin{pmatrix}H_m(x)\\ 0\end{pmatrix}dx = \frac{1}{2}\int_{-\infty}^{\infty}H_n(x)H_m(x)e^{-x^2}dx = 0,$$

i.e., the orthogonality property of the classical Hermite polynomials. The same procedure (here we multiply by $s(\omega)\omega^{-n-m-2}$) but with the E_{-n}^q functions leads to the orthogonality ($n\neq m$)

$$\int_{-\infty}^{\infty}(0, \frac{1}{2}H_{n-1}(x))\begin{pmatrix}\frac{1}{2}e^{-x^2} & 0\\ 0 & 2e^{-x^2}\end{pmatrix}\begin{pmatrix}0\\ \frac{1}{2}H_{m-1}(x)\end{pmatrix}dx = \frac{1}{2}\int_{-\infty}^{\infty}H_{n-1}(x)H_{m-1}(x)e^{-x^2}dx = 0,$$

which is similar to the previous one. Notice also that, as before, we were obliged to use different scaling factors for the $E_{\pm n}^q$ functions. Finally, for the "mixed" cases $\langle E_n^q, \widetilde{E}_{-n}^q\rangle_q'$ and $\langle E_{-n}^q, \widetilde{E}_n^q\rangle_q'$ we obtain the same relation (45) and the trivial identity $0 = 0$, respectively.

6. LIMIT NONSYMMETRIC ULTRASPHERICAL POLYNOMIALS TO HERMITE POLYNOMIALS

Finally, we will consider the limit from nonsymmetric ultraspherical polynomials to Hermite polynomials. In [14] has been shown that the functions

$$E_w^k(z) = \lim_{k\to+\infty} E_w^{k,q}(z) = R_n^k(z) + \frac{k-w}{2k+1}\frac{z-z^{-1}}{2}R_{n-1}^{k+1}(z), \quad w = \pm(n+k), \tag{46}$$

satisfy the Eq.

$$XE_w^k(z) = -w\,E_w^k(z), \tag{47}$$

where X is an operator on the space of Laurent polynomials obtained, formally, from the operators (19) and (20) by putting $Y^+ = q^{1/2X}$ and taking the limit $q \to 1-$. This yields

$$X\,F(z) = -kF(z) + z\frac{d}{dz}F(z) + 2k\frac{F(z)-F(z^{-1})}{1-z^{-2}}. \tag{48}$$

The above *Cherednik* operator jointly with the operator s obtained from T given in (18)

$$(sF)(z) = \lim_{q\to 1-}(T\,F)(z) = F(z^{-1})$$

generate the graded affine Hecke algebra of type A_1 [14].

Let us now to take the limit $k \to +\infty$ in this case. As in the previous section we will work in the space of vector functions defined in (35), but now X is given by

$$X \sim \mathbf{X} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

where

$$X_{11}f_1(x) = -kf_1(x), \quad X_{12}f_2(x) = (2k+1)2xf_2(x) + 2(x^2-1)\frac{d}{dx}f_2(x),$$
$$X_{21}f_1(x) = \frac{1}{2}\frac{d}{dx}f_1(x), \quad X_{22}f_2(x) = kf_2(x),$$

and the functions $E^k_{\pm(k+n)}(z)$ are given by

$$E^k_{n+k}(z) \longmapsto \begin{pmatrix} R_n^k(x) \\ \frac{-n}{2(2k+1)}R_{n-1}^{k+1}(x) \end{pmatrix}, \quad E^k_{-n-k}(z) \longmapsto \begin{pmatrix} R_n^k(x) \\ \frac{2k+n}{2(2k+1)}R_{n-1}^{k+1}(x) \end{pmatrix}. \tag{49}$$

Let consider the limit case corresponding to the first function $E^k_{n+k}(z)$. Let $\omega = k^{-1/2}$ ($k \to \infty$, $\omega \to 0$). Let us suppose that

$$\lim_{\omega\to 0}\frac{f_1(\omega x)}{\omega^n} = \tilde{f}_1(x), \quad \lim_{\omega\to 0}\frac{f_2(\omega x)}{\omega^{n+1}} = \tilde{f}_2(x). \tag{50}$$

In the present case this is true and we have $\tilde{f}_1(x) = 2^{-n}H_n(x)$ and

$\tilde{f}_2(x) = -2^{-n-1} n H_{n-1}(x)$. Then the equations, equivalent to (47), transform into

$$[\omega^2 X_{11}]\frac{f_1(\omega x)}{\omega^n} + \omega^2[\omega X_{12}]\frac{f_2(\omega x)}{\omega^{n+1}} = -(n\omega^2 + 1)\frac{f_1(\omega x)}{\omega^n}, \tag{51}$$

and

$$[\omega X_{21}]\frac{f_1(\omega x)}{\omega^n} + \left[\omega^2 X_{22}\right]\frac{f_2(\omega x)}{\omega^{n+1}} = -(n\omega^2 + 1)\frac{f_2(\omega x)}{\omega^{n+1}}, \tag{52}$$

which in the limits $\omega \longrightarrow 0$ lead to the Eqs.

$$-\tilde{f}_1(x) = -\tilde{f}_1(x), \quad \text{or} \quad -\frac{H_n(x)}{2^n} = -\frac{H_n(x)}{2^n},$$

and

$$\frac{1}{2}\frac{d}{dx}\tilde{f}_1(x) + \tilde{f}_2(x) = -\tilde{f}_2(x), \quad \text{or} \quad \frac{d}{dx}H_n(x) = 2nH_{n-1}(x),$$

respectively. In the other case E^k_{-n-k}, we need to impose that the limits

$$\lim_{\omega \to 0} \frac{f_1(\omega x)}{\omega^n} = \hat{f}_1(x), \quad \lim_{\omega \to 0} \frac{f_2(\omega x)}{\omega^{n-1}} = \hat{f}_2(x), \tag{53}$$

exist and they are different from zero. It happens since in this case, from (49) one has $\hat{f}_1(x) = 2^{-n}H_n(x)$ and $\hat{f}_2(x) = 2^{-n} H_{n-1}(x)$. Then providing similar calculations as before we obtain instead of (51) and (52) the expressions

$$-\hat{f}_1(x) - 2\frac{d}{dx}\hat{f}_2(x) + 4x\hat{f}_2(x) = \hat{f}_1(x) \quad \text{and} \quad \hat{f}_2(x) = \hat{f}_2(x),$$

respectively. The first Eq. is equivalent to the raising operator acting on the Hermite polynomials whereas the second is the trivial identity $H_{n-1}(x) = H_{n-1}(x)$.

Remark 6.1. Notice that in this case we have obtained two trivial identities involving the Hermite polynomials and two non trivial ones. If we want to obtain only non trivial identities we can consider the limit terms for a next higher degrees of ω. For example, if we consider the terms in ω^2 in (51)-(52) we obtain, in the matrix form,

$$\begin{pmatrix} 0 & 4x - 2\frac{d}{dx} \\ \frac{1}{2}\frac{d}{dx} & 2 \end{pmatrix} \begin{pmatrix} \tilde{f}_1(x) \\ \tilde{f}_2(x) \end{pmatrix} = -n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{f}_1(x) \\ \tilde{f}_2(x) \end{pmatrix}, \tag{54}$$

that leads to $H_n(x) = 2xH_{n-1}(x) - H'_{n-1}(x)$ and $H'_n(x) = 2nH_{n-1}(x)$. A similar procedure can be done for all cases.

To conclude this section we will study the orthogonality relation. We start from the fact that the functions $E^k_{\pm(n+k)}$ are an orthogonal set of functions, i.e., they satisfy the property

$$\int_{\pi}^{\pi} E_w^k(e^{i\theta})\overline{E_v^k(e^{i\theta})}|\sin(\theta)|^{2k}\,d\theta = 0, \quad w \neq v.$$

The above relation can be rewritten in terms of the vector functions. In fact the set of vector functions defined in (49) are an orthogonal set with respect to the inner product

$$\langle F\,,\, G\rangle_k = \int_{-1}^{1} (f_1(x), f_2(x)) \begin{pmatrix} (1-x^2)^{k-1/2} & 0 \\ 0 & 4(1-x^2)^{k+1/2} \end{pmatrix} \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix} dx.$$

Let us show the results of taking limits in the both aforementioned cases E^k_{n+k} and $E^k_{-(n+k)}$. In the first case, changing $x \mapsto \omega x$, dividing by ω^{n+m+1} and then taking the limit, we find that the following orthogonality property

$$\int_{-\infty}^{\infty} (2^{-n}H_n(x), 0) \begin{pmatrix} e^{-x^2} & 0 \\ 0 & 4e^{-x^2} \end{pmatrix} \begin{pmatrix} 2^{-m}H_m(x) \\ 0 \end{pmatrix} dx = \frac{1}{2^{n+m}} \int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx = 0.$$

For the second functions, changing $x \mapsto \omega x$, dividing by ω^{n+m-1} and then taking the limit, one gets

$$\int_{-\infty}^{\infty} (0, 2^{-n+1}H_{n-1}(x)) \begin{pmatrix} e^{-x^2} & 0 \\ 0 & 4e^{-x^2} \end{pmatrix} \begin{pmatrix} 0 \\ 2^{-m+1}H_{m-1}(x) \end{pmatrix} dx$$
$$= \frac{2^4}{2^{n+m}} \int_{-\infty}^{\infty} H_{n-1}(x)H_{m-1}(x)e^{-x^2}dx = 0.$$

For the other two cases (mixed cases) we only obtain in the limit the trivial identity $0 = 0$. To get a non trivial result we need to use the second order expression in ω (see Remark 6.1). The obtained relations are the corresponding orthogonality relations for the classical Hermite polynomials.

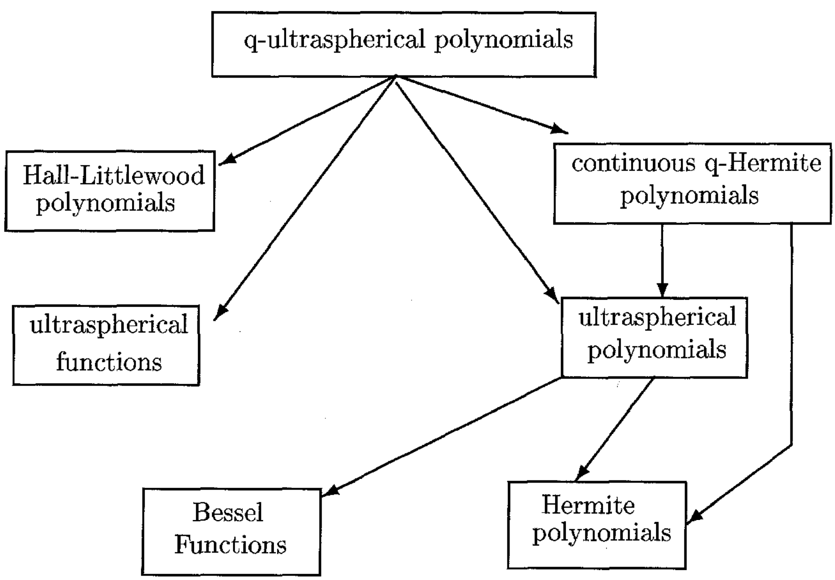

Fig. 2: Schema of the A_1 classification of special functions

To conclude this paper let us point out that from the results presented in this paper and the ones given in [14], that are summarized in figure 2, follows that the degenerating Cherednik's theory for A_1-root system leads to several well known results and identities for classical OP's such as the ultraspherical and Hermite polynomials. Here again we should mention that for the Askey-Wilson a similar study can be done as it is shown in [16]. Notice also that in the last two sections instances of vector valued polynomials appear in a very natural way. They can be considered as the rows of the corresponding (bi)orthogonal matrix polynomials (for a detailed study of matrix OP's see [6]).

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